LOCALLY DISCRETE GROUPS OF CIRCLE DIFFEOMORPHISMS

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Abstract. We describe the theory of locally discrete groups of circle diffeomorphisms developed in the last years. These are lecture notes for a 3h mini-course given during the workshop "Groupes de surfaces et difféomorphismes du cercle", held in Lyon, December 2018. Many thanks go to the organizers.

1. Preliminaries

1.1. **Topological dynamics.** Let us start recalling fundamental facts about groups acting on the circle [11, 17]. We will assume that all homeomorphisms preserve the orientation. The circle $\mathbb{S}^1$ is identified with the Euclidean torus $\mathbb{R}/\mathbb{Z}$. We denote by $\mathsf{Homeo}_+(\mathbb{S}^1)$ the group of all orientation preserving circle homeomorphisms, and by $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ the group of orientation preserving homeomorphisms of the real line $\mathbb{R}$, which commute with the group of integer translations $\mathbb{Z}$. This identifies with the universal cover of $\mathsf{Homeo}_+(\mathbb{S}^1)$ and is actually a central extension:

$$0 \to \mathbb{Z} \to \mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathsf{Homeo}_+(\mathbb{S}^1) \to 1.$$

Given $f \in \mathsf{Homeo}_+(\mathbb{S}^1)$, the *rotation number* $\rho(f) \in \mathbb{S}^1$ is defined as the limit

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \pmod{\mathbb{Z}},$$

where $F \in \mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is any lift of f and $x \in \mathbb{R}$ is any point. The rotation number is a semi-conjugacy invariant for f. It is rational of reduced fraction p/q, if and only if f has a periodic orbit of period q (and $F^q(x) = x + p$ for every $x \in \mathbb{R}$). It is irrational $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ if and only if f is semi-conjugate to the irrational rotation $R_\alpha : x \mapsto x + \alpha$. Here by *semi-conjugacy*, we mean that there exists a continuous monotone function $h : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1 such that $hf = R_\alpha h$ (h need not be invertible, that is, we are allowed to collapse f-orbits).

Let $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ be a finitely generated subgroup. A subset $\Lambda \subset \mathbb{S}^1$ is a *minimal invariant subset* if Λ is closed, non-empty and G-invariant, and minimal with respect to inclusion. When Λ is not a finite G-orbit, then Λ is unique, and can only be the whole circle or a Cantor set. In these notes, we say that a subgroup $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ is *elementary* if it admits an invariant Borel probability measure μ on $\mathbb{S}^1$. Observe that if G admits no finite orbits, then G is actually semi-conjugate to a subgroup of the group of rotations (which is actually the image of the function rotation number ρ, which defines a homomorphism in the case of an invariant probability measure). When $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ is non-elementary then there are *local exponential contractions*: for every $x \in \mathbb{S}^1$ there exists a neighborhood I_x such that a "typical" long composition of generators contracts the size $|I_x|$ at exponential rate (this can be made precise in probabilistic terms [3, 14]).

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1.2. **Hyperbolic geometry.** The group $\mathsf{PSL}(2,\mathbb{R})$ is naturally a subgroup of $\mathsf{Homeo}_+(\mathbb{S}^1)$, when considered as the group of (orientation preserving) isometries of the hyperbolic space $\mathbb{H}$ (identify the circle $\mathbb{S}^1$ with the boundary $\partial\mathbb{H}$). Observe that, with this identification, $\mathsf{PSL}(2,\mathbb{R})$ is actually a subgroup of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$, the group of *real-analytic* circle diffeomorphisms. Let us recall some classical terminology. An element $h\in\mathsf{PSL}(2,\mathbb{R})$ is *elliptic*, *parabolic* or *hyperbolic* if it has resp; 0, 1 or 2 fixed points. In the hyperbolic case, one fixed point is attracting, the other one is repelling (this is usually called a *North-South* dynamics). An elementary subgroup $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is conjugate in $\mathsf{PSL}(2,\mathbb{R})$ either to a subgroup of rotations $\mathsf{SO}(2)$ or to a subgroup of affine transformations $\mathsf{Aff}_+(\mathbb{R})$. A finitely generated subgroup $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is a *Fuchsian group* if it is discrete in the C^0 topology (by the Cauchy inequalities, all C^k topologies on $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ are equivalent). Elementary Fuchsian groups are virtually cyclic. Fuchsian groups whose minimal set is the whole circle, are called *lattices* (as in this case the quotient $\mathbb{H}/\Gamma$ has finite volume). A lattice is *uniform* (or *cocompact*) if the quotient $\mathbb{H}/\Gamma$ is compact. In general, the quotient $\mathbb{H}/\Gamma$ is a surface of finite type. Dynamically, cocompact lattices are characterized by the property that their actions on $\mathbb{S}^1\cong\partial\mathbb{H}$ are *expanding*: for every $x\in\mathbb{S}^1$ there exists an element $h\in\Gamma$ such that $h'(x)>1$. Moreover, every element $h\in\Gamma$ is either hyperbolic or elliptic (of finite order). When $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is a non-uniform lattice, there are only finitely many orbits of points where the expanding property fails, and every such point has cyclic stabilizer, generated by a parabolic element. Geometrically, these correspond to *cusps* in the quotient $\mathbb{H}/\Gamma$. At the level of group structure, a cocompact lattice is isomorphic to a *closed surface group*, whereas any other Fuchsian group is *virtually free* (ie, it contains a free subgroup of finite index). It is a classical fact that every non-elementary, non-discrete, finitely generated subgroup of $\mathsf{PSL}(2,\mathbb{R})$ contains an elliptic element of infinite order, and this element is conjugate in $\mathsf{PSL}(2,\mathbb{R})$ to an irrational rotation $R_\alpha\in\mathsf{SO}(2)$.

The topological conjugacy class of subgroups G of $\mathsf{PSL}(2,\mathbb{R})$ is characterized in dynamical terms by important works of Tukia, Casson-Jungreis, Gabai [4, 9, 20]. This is the so-called *convergence property*: for every sequence $\{g_n\}$ of distinct elements in G, such that $\{g_{n_k}\}$ is not equicontinuous, there exists a subsequence $\{g_{n_k}\}$ and points $a,b\in\mathbb{S}^1$ such that the sequence of restrictions $g_{n_k}|_{\mathbb{S}^1\setminus\{a\}}$ converges uniformly to the constant map b.

Subgroups of $\mathsf{PSL}(2,\mathbb{R})$ are very classical objects. In general, if G is a subgroup contained in a subgroup of $\mathsf{Homeo}_+(\mathbb{R})$ which is a Lie group, then G is contained in the group $\mathsf{PSL}^{(k)}(2,\mathbb{R})$, for some $k\geq 1$, the *k-fold central extension* of $\mathsf{PSL}(2,\mathbb{R})$:

$$0\to\mathbb{Z}_k\to\mathsf{PSL}^{(k)}(2,\mathbb{R})\to\mathsf{PSL}(2,\mathbb{R})\to 1.$$

Observe that apart from $k=1,2$, these are not linear groups. Subgroups of $\mathsf{PSL}^{(k)}(2,\mathbb{R})$ can be described essentially in the same way.

2. Groups of real-analytic circle diffeomorphisms

Going beyond the discussion in the previous paragraph, we would like to describe finitely generated subgroups of $\mathsf{Homeo}_+(\mathbb{S}^1)$. In this generality, this is a highly difficult task. Observe that *regularity matters*: in a recent beautiful work [13], Kim and Koberda proved that for any $r\geq 1$, there is a finitely generated subgroup $G_r\subset\mathsf{Diff}^r_+(\mathbb{S}^1)$ which is not isomorphic to any subgroup of $\mathsf{Diff}^s_+(\mathbb{S}^1)$, for any $s>r$. Here we will simply focus on subgroups $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$, hoping that the very strong regularity will be enough to grasp a good picture of the situation.

2.1. **Elementary subgroups.** Let $G\subset\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated elementary subgroup. If it has no finite orbit, then G is topologically conjugate to a group of rotations (here we make use of Denjoy theorem), and more precise statements about regularity of the conjugacy are the duty of Herman-Yoccoz theory, that we won't discuss here. If G has finite orbits, then there is a finite index subgroup H with fixed points. Moreover (and this is where real-analytic regularity comes strongly into play), every non-trivial element (and hence H) can only have finitely many fixed points.

2.2. **Non-elementary subgroups.** Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a non-elementary, finitely generated subgroup. We denote by Λ its minimal invariant subset (either a Cantor set or the whole circle). One of the consequences of the local exponential contractions we mentioned before is that G contains elements with hyperbolic fixed points in Λ:

Theorem 2.1 (Sacksteder's theorem)**.** *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a non-elementary, finitely generated subgroup and $\Lambda \subset \mathbb{S}^1$ its minimal invariant subset. Then for every open interval U intersecting Λ, there exists $p \in U \cap \Lambda$ and $g \in G$ such that $g(p) = p$ and $g(p) \neq 1$.*

We will need also the following fundamental result due to Hector, which unfortunately has no satisfactory analogue in lower regularity (*this issue was pointed out in* [7] *in the 70s and is fundamental to go beyond real-analytic regularity*).

Theorem 2.2 (Hector's lemma)**.** *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated subgroup and preserving a minimal Cantor set $\Lambda \subset \mathbb{S}^1$. Let J be a connected component of the complement $\mathbb{S}^1 \setminus \Lambda$. Then the stabilizer $\mathsf{Stab}_G(J)$ is infinite cyclic.*

Under the assumption of minimal invariant Cantor set, we have an important further result [10].

Theorem 2.3 (Ghys's theorem)**.** *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated subgroup and preserving a minimal Cantor set Λ. Then G is virtually free.*

2.3. **Local flows.** Let us first consider in more detail elementary subgroups. After the previous discussion, we can suppose that the group has fixed points and we can restrict the attention to a maximal interval without fixed points. That is, we want to describe finitely generated subgroups $H \subset \mathsf{Diff}^\omega_+(I)$ of the closed interval $I = [0,1]$ with no global fixed point in $(0,1)$. Because of real-analytic regularity, such a group H is completely determined by its image in the group of real-analytic germs $H_0 \subset \mathcal{G}^\omega_+(\mathbb{R},0)$.

Take a non-trivial germ $f \in \mathcal{G}^\omega_+(\mathbb{R},0)$ which is k-flat, for some $k \geq 0$: $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$, $a_{k+1} \neq 0$. The germ f is the time 1 of the flow of a formal vector field χ at 0, and the time t map f^t $(t \in \mathbb{R})$ writes $f^t(z) = z + ta_{k+1}z^{k+1} + O(z^{k+2})$. We claim that every germ g commuting with f belongs to this flow. For this, write the commutation relation $fg = gf$ as a system of equations of the coefficients of f and g, then one sees that g is completely determined by its $(k+1)$-th coefficient, which must equal ta_{k+1} for some t. Proceeding in this way, one succeeds in classifying solvable groups of germs [12, 16].

What happens for non-solvable groups of germs? Let us study one particular example.

Example 2.4. Let $f \in \mathcal{G}^\omega_+(\mathbb{R},0)$ be a germ with non-trivial linear part, and choose coordinates (after Poincaré-Kœnigs linearization theorem) so that $f(z) = \lambda z$. Take another germ $h \in \mathcal{G}^\omega_+(\mathbb{R},0)$ which is k-flat, $k \geq 1$, write $h(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$. For integers $m, n \in \mathbb{Z}$ we have $h^m(z) = z + ma_{k+1}z^{k+1} + O(z^{k+2})$ and

$$f^{-n}h^m f^n(z) = z + \lambda^{nk} m a_{k+1} z^{k+1} + O(\lambda^{n(k+1)} z^{k+2}).$$

Assume without loss of generality $\lambda < 1$, then for every fixed $m \in \mathbb{Z}$, the sequence $f^{-n}h^m f^n$ converges to the identity as $n \to \infty$. Moreover, for fixed t, we can choose a sequence $\{m_n\}$

such that $\lambda^{nk} m_n \to t$ as $n \to \infty$, so that $f^{-n} h^{m_n} f^n \to z + t a_{k+1} z^{k+1} + \ldots$. In other words, the group generated by f, g contains a *local flow* in its closure.

In the general case, the picture is pretty close. Let f be arbitrary; take a germ $g \in \mathcal{G}_+^\omega(\mathbb{R}, 0)$ which does not commute with f, nor does f commute with the commutator $h = [f, g]$. Then the commutator $h = [f, g]$ is k'-flat for some $k' > k$. Up to taking the inverse of f, we can assume that $a_{k+1} > 0$. Then the sequence of conjugates $f^n h f^{-n}$ converges to id as $n \to \infty$. In [16], Nakai proved that there exists an appropriate rescaling $\lambda_n(f^n h f^{-n} - id)$ which converges to a vector field $\chi_1 = \chi(f, h)$ and the flow of the vector field χ_1 belongs to the closure of the set of conjugates $\{f^n h^m f^{-n}\}_{n,m}$. In the same way, there is a limit vector field $\chi_2 = \chi(h, [f, h])$ verifying the analogous property. Moreover, the two vector fields χ_1, χ_2 are linearly independent, and are preserved under topological conjugacy. In conclusion, the dynamics of the group $\langle f, g \rangle$ on a left neighborhood of 0 is very rich, in the sense that every orbit is dense and it is well described by *local flows* which are a topological invariant of the group.

The work of Nakai was later extended by Rebelo [19] to the case of non-elementary subgroups. For this, one has to assume that G contains elements sufficiently close to the identity, more precisely, one wants G to be *non locally discrete* in the following sense:

Definition 2.5. Let $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ be a subgroup and $x \in \mathbb{S}^1$ be a point. One says that G is *non locally discrete at* x (in the C^1 topology) if there exists a neighborhood I of x and a sequence of elements $\{g_n\} \subset G$ such that $g_n|_I \to id|_I$ in the C^1 topology. If G is non locally discrete at every point of the minimal invariant subset Λ, then one simply says that G is *non locally discrete*.

Observe that by minimality of the action of G on Λ, if G is non locally discrete at a point $x \in \Lambda$, then it is everywhere non locally discrete in Λ. Moreover, Theorem 2.2 implies that if Λ is a Cantor set, then G must be locally discrete at points $x \notin \Lambda$. In fact, the following holds:

Theorem 2.6 (Rebelo)**.** *Let $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ be a finitely generated non-elementary subgroup which is non locally discrete. Then the action of G on $\mathbb{S}^1$ is minimal, that is, $\Lambda = \mathbb{S}^1$.*

The key observation by Rebelo is that such a group contains local flows in the closure, essentially by the same argument in Example 2.4. For this the element f is replaced by a map with an hyperbolic fixed point $p \in I \cap \Lambda$ given by Theorem 2.1, and one plays with a sequence of elements g_n whose restriction to I converges to the identity in the C^1 topology. It is at this point that the C^1 topology is needed: the elements g_n do not necessarily fix the point p, and one needs to control their powers. Rebelo proves that for sufficiently small $|t| < \varepsilon$, there exist sequences $k_n, \ell_n(t)$ such that $f^{-k_n} g_n^{\ell_n(t)} f^{k_n}$ converges to the time t of a locally defined flow.

Non locally discrete groups have been extensively studied by Rebelo and collaborators. Let us point out one further consequence of local flows:

Proposition 2.7. *Let $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ be a finitely generated non-elementary subgroup which is non locally discrete. Then the action of G on $\mathbb{S}^1$ is expanding: for every $x \in \mathbb{S}^1$ there exists $g \in G$ such that $g'(x) > 1$.*

Proof. Indeed, suppose that for such a group there exists a point x such that $g'(x) \leq 1$ for every $g \in G$. Let I be a neighborhood of x on which local flows are defined, and take a point $p \in I$ and $h \in G$ such that $h(p) = p$, $h'(p) \leq 1$, given by Theorem 2.1. Then using the local flow, we conjugate h to an element h_ε having a hyperbolic fixed point p_ε which is ε-close to x, with derivative $h_\varepsilon'(p_\varepsilon) = h'(p)$ and, which is more important, we keep control on its derivative around p_ε because we conjugate by elements which are close to the identity. By this control on derivative, there exists $\varepsilon > 0$ such that $h'_\varepsilon(p) > 1$, a contradiction. □

2.4. **Locally discrete subgroups.** The discussion in the previous paragraph indicates that the notion of discreteness is not well-suited for treating groups of real-analytic diffeomorphisms, but rather local discreteness is the appropriate property. Locally discrete groups are nowadays a subject of very active investigation.

Conjecture 2.8. *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated, locally discrete subgroup. One of the following holds:*

(1) either G is C^ω conjugate to a cocompact discrete subgroup of $\mathsf{PSL}^{(k)}(2,\mathbb{R})$, for some $k \geq 1$, or
(2) G is virtually free, and the action is described by a ping-pong partition (in the sense of Definition 3.2).

This conjecture has been validated for groups G which are virtually free [1,6], with one end, finitely presented and bounded torsion [8] and groups with infinitely many ends [2]. The first possibility has been completely described [5]:

Theorem 2.9 (Deroin)**.** *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated, locally discrete subgroup. Then G is C^ω conjugate to a cocompact discrete subgroup of $\mathsf{PSL}^{(k)}(2,\mathbb{R})$, for some $k \geq 1$, if and only if the action of G on $\mathbb{S}^1$ is expanding.*

The strategy for working on this conjecture, besides the work of Deroin, is to understand the set of *non-expandable points* $\mathsf{NE}(G) = \{x \in \Lambda : g'(x) \leq 1 \text{ for every } g \in G\}$. In the case of Fuchsian groups, we observed that these points are related to the cusps of the quotient surface. Similarly here one wants to *prove that every $x \in \mathsf{NE}(G)$ is a parabolic fixed point for some element in G.* Control of affine distortion is the key: given an interval J and a C^1 map $f : \mathbb{S}^1 \to \mathbb{S}^1$, one defines the *distortion coefficient*

$$\varkappa(f;J) = \sup_{x,y\in J} \log \frac{f'(x)}{f'(y)}.$$

This is a classical way to measure how f deviates from being an affine map: $\varkappa(f;J) \geq 0$ and $\varkappa(f;J) = 0$ if and only if f is affine on J. Moreover, this coefficient has the nice feature to well-behave under composition:

$$\varkappa(gf;J) \leq \varkappa(g;f(J)) + \varkappa(f;J). \tag{1}$$

$$\varkappa(f;I \cup J) \leq \varkappa(f;I) + \varkappa(f;J). \tag{2}$$

The second important property is that $\varkappa(f;J)$ is a continuous Lipschitz function with respect to J: $\varkappa(f;J) \leq \sup_{\mathbb{S}^1} |(\log f')'| \cdot |J|$. By the mean value theorem, the distortion coefficient allows to replace the size of an interval with the derivative at one given point, and vice versa: for every $x_0 \in J$ one has

$$e^{-\varkappa(f;J)} f'(x_0) \leq \frac{|f(J)|}{|J|} \leq e^{\varkappa(f;J)} f'(x_0). \tag{3}$$

Let us explain the approach that has been successful so far [2,6,8]. Take a point $x_0 \in \mathsf{NE}(G)$ and observe that its G-orbit is dense in Λ. This means that there are sequence of elements $\{g_n\}$ and such that $g_n(x_0) - x_0 = \varepsilon_n > 0$, for some sequence $\varepsilon_n \searrow 0$. If one can obtain a good control of distortion for these maps on the intervals $J_n = [x_0, g_n(x_0)]$, then one must have that the derivative of g_n is close to 1 on J_n (the conditions $g'_n(x_0), (g_n^{-1})'(x_0) \leq 1$ imply that there exists a point $y_n \in J_n$ such that $g_n'(y_n) = 1$), and so g_n is close to the identity on J_n (because $g_n(x_0)$ is ε_n-close to x_0). Typically one would like to take as g_n the element in the ball of radius n in G (with respect to some generating system), giving the closest return of x_0, but in practice the choice of g_n has to be adapted to the case under consideration.

3. DKN ping-pong

3.1. **Free groups.** We recall the main construction in [6] for free groups. We denote by G a rank-n free group. We choose S_0 a system of free generators for G, and write $S = S_0 \cup S_0^{-1}$. We denote by $\|\cdot\|$ the word norm defined by S. Recall that any element g in G may be written in unique way

$$g = g_\ell \cdots g_1, \quad g_i \in S,$$

with the property that if $g_i = s$ then $g_{i+1} \neq s^{-1}$. This is called the *normal form* of g. For any $s \in S$, we define the set

$$W_s := \{g \in G \mid g = g_\ell \cdots g_1 \text{ in normal form, with } g_1 = s\}.$$

If X denotes the Cayley graph of G with respect to the generating set S (which is a $2n$-regular tree), then the sets W_s are exactly the $2n$ connected components of $X \setminus \{id\}$, with W_s being the connected component containing s. It is easy to see that the generators S play ping-pong with the sets W_s: for any $s \in S$, we have $(X \setminus W_{s^{-1}})\, s \subset W_s$ (we consider the right action of G on X). Using the action on the circle, we can push this ping-pong partition of the Cayley graph of G to a partition of the circle into open intervals with very nice dynamical properties. Given $s \in S$, we define

$$U_s := \left\{ x \in \mathbb{S}^1 \mid \exists \text{ neighbourhood } I_x \ni x \text{ s.t. } \lim_{n\to\infty} \sup_{g \notin W_s, \|g\| \geq n} |g(I_x)| = 0 \right\}. \tag{4}$$

In [6] it is proved the following:

Theorem 3.1 (Deroin, Kleptsyn, and Navas)**.** *Let $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ be a finitely generated, locally discrete, free group of real-analytic circle diffeomorphisms. Let S_0 be a system of free generators for G and write $S = S_0 \cup S_0^{-1}$. Consider the collection $\{U_s\}_{s\in S}$ defined in (4). We have:*

(1) every U_s is open;
(2) every U_s is the union of finitely many intervals;
(3) any two different U_s have empty intersection inside the minimal invariant set Λ_G;
(4) the union of the U_s covers all but finitely many points of Λ_G;
(5) if $s \in S$, $t \neq s$ then $s(U_t) \subset U_{s^{-1}}$.

Definition 3.2. Let $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ be a finitely generated, free group of circle homeomorphisms and let $S = S_0 \cup S_0^{-1}$ be a symmetric free generating set. A collection $\{U_s\}_{s\in S}$ of subset of $\mathbb{S}^1$ is a *ping-pong partition* for (G, S) if it verifies conditions 1. through 5. in Theorem 3.1.

For $s \in S$, denote by U_s^* the subset of U_s which is the union of the connected components of U_s intersecting Λ_G. The *skeleton* of the ping-pong partition is the data consisting of

(1) The cyclic order in $\mathbb{S}^1$ of the intersection of connected components of $\bigcup_{s\in S} U_s$ with Λ_G, and
(2) For each $s \in S$, the assignment of connected components

$$\lambda_s : \pi_0\left(\bigcup_{t\in S\setminus\{s\}} U_t^*\right) \to \pi_0\left(U_{s^{-1}}\right)$$

induced by the action.

Remark 3.3. In [6] the definition of the sets U_s (there called $\widetilde{\mathcal{M}_\gamma}$) is slightly different based on a control on the sum of derivatives along geodesics in the group. Here the definition that we adopt is simply topological, as we consider how neighbourhoods are contracted along geodesics in the group. This difference in the definition leads to different sets: one can show that U_s contains the corresponding $\widetilde{\mathcal{M}_s}$, and the complement $U_s \setminus \widetilde{\mathcal{M}_s}$ is a finite number of points.

Even with the different definition, the proof of Theorem 3.1 proceeds as in [6]. The hardest part is to prove that property 2., which is Lemma 3.30 in [6]. Property 1. is a direct consequence of the definition, 3. is an easy consequence of Theorem 2.1, 4. can be obtained from minimality (of the pieces of orbits $W_s(x)$, $x \in \mathbb{S}^1$, $s \in S$) and the other properties, as soon as one knows that at least one U_s is *non-empty* (this is not so difficult in the case Λ is a Cantor set, but it is a highly non-trivial statement for minimal actions, which requires to understand the points in $\mathsf{NE}(G)$ as explained before). The ping-pong property 5. comes directly from the definition of U_s and the fact that the generators S play ping-pong with the sets W_s.

Before sketching the proof of property 2., let us state a classical result (see [15, Theorem 4.7]) explaining why ping-pong partitions are important. For this, we first need the following:

Definition 3.4. Let $\rho_\nu : (G, S) \to \mathsf{Homeo}_+(\mathbb{S}^1)$, $\nu = 1, 2$, be two injective representations of a finitely generated, free group with a marked symmetric free generating set $S = S_0 \cup S_0^{-1}$. Let $\{U_s^\nu\}_{s\in S}$, be a ping-pong partition for $\rho_\nu(G, S)$, for $\nu = 1, 2$. We say that the two partitions are *equivalent* if they have the same skeleton.

Proposition 3.5. *Let $\rho_\nu : (G, S) \to \mathsf{Homeo}_+(\mathbb{S}^1)$, $\nu = 1, 2$, be two injective representations of a finitely generated, free group with a marked symmetric free generating set $S = S_0 \cup S_0^{-1}$. Suppose that the actions on $\mathbb{S}^1$ have equivalent ping-pong partitions. Then the actions are semi-conjugate.*

3.2. **Proof of Theorem 3.1.** In the course of the proof, we will make the simplifying assumption that the action is minimal. Given an open set U_s, fix one of its connected components $I = (x_-, x_+)$.

Definition 3.6. An element $g \notin W_s \cup \{id\}$ is *wandering* if, writing $g = g_n \cdots g_1$ in normal form, one has that the intervals $\{g_k \cdots g_1(I)\}_{k=0}^{n-1}$ are all disjoint. We say also that g is a *first-return* if $g(I) \cap I \neq \emptyset$.

Lemma 3.7.

a) If g, h are two distinct wandering elements, then $g(I) \cap h(I) = \emptyset$.
b) If $g = g_n \cdots g_1$ is a first return, then $g_n = s^{-1}$ and $g(I) \subset I$.
c) There exists $C > 0$ such that for every wandering element g one has $\varkappa(g; I) \leq C$.
d) There exists $\delta > 0$ and $C' > 0$ such that for every admissible element g one has $\varkappa(g; \tilde{I}) \leq C'$, where $\tilde{I}$ denotes the δ-neighborhood od I.

Proof. Write $g = g_n \cdots g_1$ and $h = h_m \cdots h_1$. By the ping-pong relation 5., we have $g(I) \subset U_{g_n^{-1}}$ and $h(I) \subset U_{h_m^{-1}}$. So the images are disjoint, unless $g_n = h_m$. If this happens, consider $g' = g_{n-1} \cdots g_1$ and $h' = h_{m-1} \cdots h_1$ and repeat the argument until one of the two elements is trivial (as $g \neq h$ the other element is non-trivial). Then statement reduces to the condition of being wandering. This proves a). The second statement b) is a consequence of the ping-pong relation 5. and the fact that I is a connected component. The third statement is a consequence of a) applied to the sequence of wandering elements $\{g_k \cdots g_1\}_{k=1}^n$ and the sub-additivity of the distortion coefficient (1). The statement d) is more subtle, although it is a very general statement; it requires to apply (2) and (3) inside an inductive argument. □

3.3. **First-returns.** We recall the following definition:

Definition 3.8. A map g is a *uniform contraction* on an interval J if there exists $0 < \rho < 1$ such that for any subinterval $E \subset J$ one has $|g(E)| \leq \rho|E|$. In this case one says that g is a uniform contraction of ratio $\leq \rho$.

Lemma 3.9. *Let $\tilde{I}$ be the δ-neighborhood of I as in Lemma 3.7.d and let R denote the rightmost connected component of $\tilde{I} \setminus I$. If g is a first return which is a uniform contraction on $I \cup R$, then for every $I \subset I' \subset I \cup R$ one has $g(I') \subset I'$.*

Proof. Indeed, by Lemma 3.7, one has $g(I) \subset I$. Let R' be the rightmost connected component of $I' \setminus I$. The image $g(R')$ is adjacent to the right of $g(I)$ and has length $|g(R')| < |R'|$. Therefore, $g(R')$ cannot trespass the rightmost point of I'. □

Lemma 3.10. *For any $0 < \rho < 1$, all but finitely many wandering elements are uniform contractions on $\tilde{I}$, of ratio $\leq \rho$.*

Proof. Fix $\varepsilon \in (0, 1)$. By Lemma 3.7, there is only a finite number of first-returns g such that $|g(I)| \geq \varepsilon|I|$. Consider a first return g such that $|g(I)| < \varepsilon|I|$. Then there exists a point $x_0 \in I$ such that $g'(x_0) < \varepsilon$ and hence, by Lemma 3.7.d, we have that for any $x \in \tilde{I}$, $g'(x) \leq e^{C'}\varepsilon$. In particular, if $\varepsilon < \rho e^{-C'}$, such a first-return is a uniform contraction on $\tilde{I}$, of ratio $\leq \rho$. □

In the following, given an element $g \in G$, we can write in a unique way g as the product

$$g = f\pi_\ell \cdots \pi_0,$$

where the π_k are first return, and f is a wandering element. We call this the *first-return decomposition* of g.

Proposition 3.11 ([6], Lemma 3.23)**.** *There exist first-returns g_- and g_+ that fix respectively the left and right endpoints x_- and x_+ of I.*

Proof. After Lemma 3.10, there is only a finite collection of first-returns $g_1, \ldots, g_m$ which are not uniform contractions on $\tilde{I}$ of ratio $\leq \frac{1}{2}$. We want to show that there is some g_i fixing x_+. We shall argue by contradiction; supposing that this is not the case, there exists a compact subinterval $J \subset I$ which contains a neighbourhood of all the points $g_i(x_+)$'s. Let R be a neighbourhood of x_+ which is contained in $\tilde{I}$, and such that $g_i(R) \subset J$ for all $i = 1, \ldots, m$. Observe that this condition and Lemma 3.9 imply that $g(R) \subset I \cup R$ for every first-return g.

We want to prove that, if none of the g_i's fixes x_+, then

$$\lim_{n\to\infty} \sup_{g \notin W_s, \|g\| \geq n} |g(R)| = 0,$$

which will imply that $x_+ \in I$, which is absurd. We fix $\varepsilon > 0$ and we suppose that there exists a sequence of elements $h_n \notin W_s$ such that for every $n \in \mathbb{N}$,

$$\|h_n\| \geq n \quad \text{and} \quad |h_n(R)| \geq \varepsilon. \tag{5}$$

After Lemma 3.10, we can suppose that none of h_n's is a wandering element: choosing $\rho < \frac{\varepsilon}{|R|}$, for all but finitely many wandering elements g we have $|g(R)| \leq \rho|R| < \varepsilon$.

Let $h \notin W_s$ be a non-wandering element which satisfies (5). Let us consider the first-return decomposition

$$h = f\pi_\ell \cdots \pi_0.$$

If none of the π_i's is one of the fixed first-returns $g_1, \ldots, g_m$, then

$$|h(R)| \le \frac{|f(R)|}{2^\ell} \le \frac{1}{2^\ell}.$$

In this case ℓ must be bounded, and proceeding as before, we see that only finitely many such h can satisfy (5).

Next, suppose that there is some π_i which is the first of the first-returns $g_1, \ldots, g_m$ that appears in the first-return decomposition of h. Let us write $h = h_1 \pi_i h_2$, where no g_i appears in h_2. One can check that

$$h(R) = h_1 \pi_i h_2(R) \subset h_1(J).$$

Using compactness of J, we know that for at most finitely many elements h_1 we have

$$|h_1(J)| \ge \varepsilon.$$

Moreover, discarding finitely many h_2 we can assume that $h_1(R)$ is arbitrarily small, so that by control of distortion also $\pi_i h_2(R) \subset J$ is. By compactness of J, we can cover it by finitely many intervals $J_1, \ldots, J_p$ satisfying

$$|h_2(J_k)| < \varepsilon, \quad \forall h_2 \notin W_s,$$

and such that any interval of length $\le \delta$ (for some $\delta > 0$) is contained in one of the J_k. This gives the desired contradiction. □

Lemma 3.12 ([6], Lemma 3.29)**.** *With the previous notations, one has $g'_\pm(x_\pm) = 1$ and they are topological contractions of I.*

Proof. It is enough to observe that they are topological expansions on the other side of $x_\pm$ respectively (this is because otherwise we will have control on contractions at points $x_\pm$). □

Lemma 3.13 ([6], Lemma 3.24)**.** *With the previous notations, the only fixed point of $g_\pm$ in $\overline{I}$ is $x_\pm$. In particular one has $g_- \neq g_+$.*

Proof. If g_+ fixes an interval in I, by Lemma 3.7 there would be a wandering interval (for the images under $g \notin W_s$), contradicting minimality of $W_t(x)$ for $x \in \mathbb{S}^1$, $t \in S$. □

3.4. A finite number of connected components.

Lemma 3.14 ([6], Lemma 3.23)**.** *Each U_k has only a finite number of connected components.*

Proof. Define a dynamics of intervals: given I and the corresponding g_+, write $g_+ = g_k \cdots g_1$, and say that $\Phi_+(I)$ is the connected component of the U_s which contains $g_1(I)$. This gives decomposes the intervals of U_s into cycles, which are either disjoint or coincide. Similarly one defines Φ_-.

Fixing one I, we can assume $|g_+(I)| \le \frac{1}{2}|I|$, so that by control of distortion we have $\varkappa(g_+; I) \le \log 2$ and hence $\sum |g_i \cdots g_1(I)| \ge \frac{\log 2}{C}$. This implies that there are finitely many cycles. □

3.5. **Virtually free groups.** According to Conjecture 2.8, we would like to have a good combinatorial understanding of all locally discrete virtually free subgroups $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$. This is the object of an ongoing work and the main difference, compared to [6], is that we have to make extensive use of Bass-Serre theory. Observe that the algebraic structure of virtually free groups acting on the circle is well-understood:

Theorem 3.15. *Let $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ be a virtually free group. Then G is a finite cyclic extension of a free group: there exists a free subgroup $F \subset G$ of index $m \in \mathbb{N}$ such that G fits into a short exact sequence*

$$1 \to F \to G \to \mathbb{Z}_m \to 0.$$

Conversely, any finite cyclic extension of a free group can be realized as a locally discrete group of real-analytic circle diffeomorphisms.

For establishing an analogue of Theorem 3.1 and Proposition 3.5, we have to develop a good notion of ping-pong for virtually free groups. More generally, we obtain a ping-pong lemma for fundamental groups of graphs of groups. As the definition is quite involved (one has to verify a list of 9 conditions), we won't go into details here. One of the nice features of the ping-pong partition of the circle that we obtain is that it well-behaves when considering finite-index subgroups, so that we can actually rely on the work done for free groups in the course of the proof.

3.6. **Thurston–Tsuboi group.** As a conclusion, let us mention a very interesting subgroup of $\mathsf{Diff}_+^\omega(\mathbb{S}^1)$ appearing in early works of Thurston and Tsuboi. Every $\mathsf{PSL}^{(k)}(2,\mathbb{R})$ identifies as a subgroup of $\mathsf{Diff}_+^\omega(\mathbb{S}^1)$, and the identification is determined by the way that $\mathsf{SO}(2) \subset \mathsf{PSL}^{(k)}(2,\mathbb{R})$ acts on $\mathbb{S}^1$. As $\mathsf{SO}(2) \subset \mathsf{PSL}^{(k)}(2,\mathbb{R})$ for any $k \in \mathbb{N}$, this gives a consistent way to make all the $\mathsf{PSL}^{(k)}(2,\mathbb{R})$ act simultaneously on $\mathbb{S}^1$. One can consider the group $\mathbb{G}$, defined as the amalgamated product of all $\mathsf{PSL}^{(k)}(2,\mathbb{R})$ over $\mathsf{SO}(2)$. The subgroup $\mathbb{G}$ is dense in $\mathsf{Diff}_+^\omega(\mathbb{S}^1)$. By taking a bunch of elements in $\mathbb{G}$, one can easily construct many examples of virtually free groups which are locally discrete, but are not contained in any conjugate of $\mathsf{PSL}^{(k)}(2,\mathbb{R})$. Moreover, stability of ping-pong partitions imply that *every* locally discrete virtually free group can be realized as a discrete subgroup of $\mathbb{G}$. This groups seems more of geometric nature than the whole $\mathsf{Diff}_+^\omega(\mathbb{S}^1)$. Can one obtain some geometry for virtually free groups out of it?

3.7. **Lower regularity.** Ping-pong partitions can be used to define a systematic family of examples of locally discrete subgroups of $\mathsf{Diff}_+^\infty(\mathbb{S}^1)$, similar to (the C^∞ version) of Thompson's group T. Given a virtualy free group $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$, this is done by considering all diffeomorphisms of $\mathbb{S}^1$ which map one refinement of the ping-pong partition to another, locally by elements of G. Are these the only examples of locally discrete subgroups in lower regularity?

References

[1] J. Alonso, S. Alvarez, D. Malicet, C. Meniño Coton, and M. Triestino, *Maskit partitions and locally discrete groups of real-analytic circle diffeomorphisms, I: Construction.*

[2] S. Alvarez, D. Filimonov, V. Kleptsyn, D. Malicet, C. Meniño Coton, A. Navas, and M. Triestino, *Groups with infinitely many ends acting analytically on the circle*, available at arXiv:1506.03839.

[3] V. A. Antonov, *Modeling of processes of cyclic evolution type. Synchronization by a random signal*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **vyp. 2** (1984), 67–76 (Russian, with English summary). MR756386

[4] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered* 3*-manifolds*, Invent. Math. **118** (1994), no. 3, 441–456, DOI 10.1007/BF01231540. MR1296353

[5] B. Deroin, *Locally discrete expanding groups of analytic diffeomorphisms of the circle.* In preparation.

[6] B. Deroin, V.A. Kleptsyn, and A. Navas, *On the ergodic theory of free group actions by real-analytic circle diffeomorphisms*, Invent. Math. **212** (2018), no. 3, 731–779.
[7] P.R. Dippolito, *Codimension one Foliations of Closed Manifolds*, Ann. Math. (2) **107** (1978), no. 3, 403–453.
[8] D.A. Filimonov and V.A. Kleptsyn, *One-end finitely presented groups acting on the circle*, Nonlinearity **27** (2014), no. 6, 1205–1223.
[9] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) **136** (1992), no. 3, 447–510, DOI 10.2307/2946597. MR1189862
[10] É. Ghys, *Classe d'Euler et minimal exceptionnel*, Topology **26** (1987), no. 1, 93–105.
[11] ———, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), no. 3-4, 329–407. MR1876932
[12] Yu.S. Ilyashenko, *The topology of phase portraits of analytic differential equations in the complex projective plane*, Trudy Sem Petrovsky **4** (1978), 83–136.
[13] S.-h. Kim and T. Koberda, *Diffeomorphism groups of critical regularity*, available at arXiv:1711.05589.
[14] D. Malicet, *Random walks on* $\mathrm{Homeo}(S^1)$, Comm. Math. Phys. **356** (2017), no. 3, 1083–1116, DOI 10.1007/s00220-017-2996-5. MR3719548
[15] S. Matsumoto, *Basic partitions and combinations of group actions on the circle: a new approach to a theorem of Kathryn Mann*, Enseign. Math. **62** (2016), no. 1-2, 15–47, DOI 10.4171/LEM/62-1/2-4. MR3605808
[16] I. Nakai, *Separatrices for non solvable dynamics on* $\mathbb{C}, 0$, Annales de l'institut Fourier **44** (1994), no. 2, 569–599.
[17] A. Navas, *Groups of circle diffeomorphisms*, Translation of the 2007 Spanish edition, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011. MR2809110
[18] A. Navas and M. Triestino, *On the invariant distributions of* C^2 *circle diffeomorphisms of irrational rotation number*, Math. Z. **274** (2013), no. 1-2, 315–321, DOI 10.1007/s00209-012-1071-3. MR3054331
[19] J.C. Rebelo, *Ergodicity and rigidity for certain subgroups of* $\mathsf{Diff}^\omega(S^1)$, Ann. Scient. Éc. Norm. Sup. (4) **32** (1999), 433–453.
[20] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. **391** (1988), 1–54, DOI 10.1515/crll.1988.391.1. MR961162