GROUPS OF AFFINE AND PIECEWISE AFFINE HOMEOMORPHISMS

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Abstract. The translation $T : x \mapsto x+1$ and the multiplication $D : x \mapsto 2x$ generate a group of affine homeomorphisms of the real line, usually named $\mathsf{BS}(1,2)$ after Baumslag-Solitar, or dyadic affine group. This group is generated by the elements T, D, with the only relation $DTD^{-1} = T^2$. This relation is definitely good from a dynamical point of view, in the sense that for any action, the dynamics of T is conjugate to the action of the square T^2. Starting from is, the continuous actions of $\mathsf{BS}(1,2)$ on the real line are now well understood (Guelman-Liousse [11]). It turns out that they are even more rigid (Bonatti-Monteverde-Navas-Rivas [8]). Introducing a third element H, coinciding with D in restriction to the positive half-line, at with the identity in restriction to the negative half-line, on finds an interesting group acting (faithfully) C^0 but not C^1 on the line (Bonatti-Lodha-T. [7]). This group embeds in the recent examples of Monod [15], Lodha and Tatch-Moore [13] of nonamenable groups without free subgroups.

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1. Affine groups and Baumslag-Solitar groups

The group of (orientation preserving) affine transformations of the real line $\mathsf{Aff}_+(\mathbb{R})$ is isomorphic to the semi-direct product $\mathbb{R} \rtimes \mathbb{R}_+$: conjugating a translation by a homothety gives a new translation. Therefore every subgroup of $\mathsf{Aff}_+(\mathbb{R})$ is abelian or metabelian.

In this notes we shall be interested in studying actions of some finitely generated subgroups of $\mathsf{Aff}_+(\mathbb{R})$, for a twofold reason: (1) the structure of these groups is quite simple; (2) we know that they admit an action (the standard affine action) which can serve as reference action.

Example 1.1. A family of translations generate an abelian group. Similarly, a family of homotheties generate an abelian group.

Example 1.2. The translation $T : x \mapsto x + 1$ and the homothety $D : x \mapsto 2x$ generated the group $\mathsf{Aff}_+(\mathbb{Z}[\frac{1}{2}])$ of affine transformations of the ring $\mathbb{Z}[\frac{1}{2}]$. This group has the presentation $\langle D, T \mid DTD^{-1} = D^2\rangle$, so it is isomorphic to the so-called *Baumslag-Solitar group* $\mathsf{BS}(1,2)$. Replacing $x \mapsto 2x$ by $x \mapsto nx$, $n \in \mathbb{N}$, one obtains the group $\mathsf{Aff}_+(\mathbb{Z}[\frac{1}{n}]) \cong \mathsf{BS}(1,n) = \langle a, b \mid aba^{-1} = b^n\rangle$.

We can also consider the group generated by T together with the homothety $x \mapsto \lambda x$, $\lambda > 1$. The group is isomorphic to the semi-direct product $\mathbb{Z}[\lambda, \lambda^{-1}] \rtimes \mathbb{Z}$, where the right factor acts by multiplication by λ. When λ is irrational, the group is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$, whilst for $\lambda = n/m$ rational the group is a homomorphic image of the Baumslag-Solitar group $\mathsf{BS}(m,n) = \langle a, b \mid ab^m b^{-1} = b^n\rangle$. These one-relator groups where introduced by Bausmlag and Solitar in 1962 [5] to give the first examples of *non-hopfian* groups (a group G is *hopfian* if $G/N \cong G$ implies $N = \{id\}$). Indeed, $\mathsf{BS}(2,3)$ is non-hopfian, and hence highly different

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from $\mathbb{Z}[\frac{3}{2},\frac{2}{3}]\rtimes\mathbb{Z}=\mathbb{Z}[\frac{1}{6}]\rtimes\mathbb{Z}$. Therefore, it is with some abuse of language that one refers to $\mathsf{Aff}_+(\mathbb{Z}[\frac{1}{n}])$ as to a Baumslag-Solitar group... For this reason the groups $\mathsf{BS}(1,n)$ are sometimes called the *solvable*, or *affine*, Baumslag-Solitar groups.

Example 1.3. Other interesting groups of affine transformations are the *abelian-by-cyclic* groups $\Gamma_A=H\rtimes_A\mathbb{Z}$, where H is a free abelian group of translations (of finite rank), and $\mathbb{Z}$ is generated by a homothety that acts on H as a matrix $A\in\mathsf{GL}(d,\mathbb{Q})$. When A has no eigenvalue of norm 1, the group Γ_A shares many similarities with the groups $\mathsf{BS}(1,n)$.

For instance, taking d $\mathbb{Q}$-independent translations $T_i:x\mapsto x+t_i$, $i=1,\dots,d$, and a homothety $D:x\mapsto nx$, defines the group $\langle D,T_1,\dots T_d\rangle\cong\mathbb{Z}^d\rtimes_{nI_d}\mathbb{Z}$, where nI_d is the diagonal matrix $\mathsf{diag}(n,\dots,n)$.

The reason actions of (solvable) Baumslag-Solitar groups are widely studied is because of the simple presentation, given by just one relation. Moreover, as we shall see in action, the relation $ab^ma^{-1}=b^n$ has a dynamical meaning: a conjugates a power of b to another power.

One of the first relevant works in this subject is by Burslem-Wilkinson [9], where they study sufficiently regular actions of $\mathsf{BS}(1,n)$ on the circle. This was later improved by Guelman-Liousse [11], and finally by Bonatti-Monteverde-Navas-Rivas [8]. For actions on higher-dimensional manifolds, McCarthy [14] proved that C^1 perturbations of the trivial action of Γ_A are not faithful. Another example of rigidity result was obtained by Asaoka [3,4] for standard actions of Γ_A on spheres and tori, and also by Wilkinson-Xue [20] for actions on tori. Finally, planar actions of $\mathsf{BS}(1,n)$ have been investigated by several authors [1,2,12].

Our aim is to us illustrate some of the low-tech methods with a couple of examples.

Example 1.4. The group $\mathsf{BS}(1,1)$ is simply the abelian group $\mathbb{Z}^2$. We can describe all possible actions of $\langle a,b\mid[a,b]\rangle\cong\mathbb{Z}^2$ on the real line.

Suppose first the generator a has no fixed point on $\mathbb{R}$. Then b defines a homeomorphism $\overline{b}$ of the circle $\mathbb{S}^1_a:=\mathbb{R}/\langle a\rangle$. Reciprocally, any homeomorphism of $\mathbb{S}^1_a$ lifts to a homeomorphism of $\mathbb{R}$ commuting with a. Therefore the classification of actions of $\mathbb{Z}^2$ on $\mathbb{R}$, with a primitive element acting without fixed points is given by conjugacy classes of actions of $\mathbb{Z}$ on the circle. We remark that b has a well-defined *relative translation number*, $\mathsf{rot}^\sim_a(b)=\lim_{n\to\infty}\frac{b^n(x)-x}{a^n(x)-x}$, which is given by the translation number of the lift b.

Suppose now that a has fixed points, and let I be a connected component of $\mathbb{R}\setminus\mathsf{Fix}(a)$. As a and b commute, the images $b^k(I)$, $k\in\mathbb{Z}$, are also connected components of $\mathbb{R}\setminus\mathsf{Fix}(a)$. Observe that we have the property that for any connected component $I\subset\mathbb{R}\setminus\mathsf{Fix}(a)$ the images $b_k(I)$, $k\in\mathbb{Z}$, either coincide, or they are pairwise disjoint.

In the first case, $b|_I$ defines a homeomorphism of the circle $I/\langle a|_I\rangle$, as in the previous situation. In the second case, the two accumulation points of $b^k(x)$, $x\in I$ define an open interval J, which contains I. The map b has no fixed point in J, so $a|_J$ defines a homeomorphism of the circle $J/\langle b|_J\rangle$.

Given a connected component $I\subset\mathbb{R}\setminus\mathsf{Fix}(a)$ and b commuting with a, it is convenient to extend the notion of relative translation number by setting

$$\mathsf{rot}^\sim_a(b,I):=\lim_{n\to\infty}\frac{b^n(x)-x}{a^n(x)-x},\quad x\in I.$$

This relative translation number is allowed to take the values $\pm\infty$, and this happens if and only if b takes I disjointly from itself. If J is a connected component of $\mathbb{R}\setminus\mathsf{Fix}(b)$ such that

$I \cap J \neq \emptyset$, then one finds the relation

$$\mathsf{rot}^{\sim}_{a}(b, I) = \frac{1}{\mathsf{rot}^{\sim}_{b}(a, J)}.$$

(Here the reader should recognize the continued fraction algorithm.)

Example 1.5. An other interesting example is provided by the so-called *Klein group* $\mathsf{BS}(1, -1) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$, which is isomorphic to the fundamental group $\pi_1(K)$ of the Klein bottle K. Also in this case we are able to describe all the possible actions of the group on $\mathbb{R}$, and we will see that the range of possibilities will be much more restricted. For simplicity, we assume that the action of $\pi_1(K)$ on $\mathbb{R}$ has no global fixed points.

We first prove that a cannot act with fixed points. Indeed, suppose there was a fixed point $p \in \mathbb{R}$, which is not fixed by b. Without loss of generality, we can assume $b^{-1}(p) < p < b(p)$. Applying a, we get $p < ab(p)$. However $ab(p) = b^{-1}a(p) = b^{-1}(p) < p$, contradiction.

Then we prove that b must have a fixed point. If this was not the case, we could assume $x < b(x)$ for every $x \in \mathbb{R}$. Then given a point $x \in \mathbb{R}$, we would have

$$a(x) < ab(x) = b^{-1}(a(x)) < a(x),$$

which is absurd. Moreover, if p is a fixed point for b, then all the images $a^k(p)$, $k \in \mathbb{Z}$, are fixed points, as one deduces from the relation $ab = b^{-1}a$.

So, given a fixed point p for b, the interval $I = [p, a(p)]$ is preserved by b, and it corresponds to a fundamental domain for a. The relation $aba^{-1} = b^{-1}$ implies that b is completely determined by its behavior on I: for any $x \in I$,

$$b(a^k(x)) = \begin{cases} a^k b^{-1}(x) & \text{if } k \text{ odd}, \\ a^k b(x) & \text{if } k \text{ even}. \end{cases}$$

The result of all of this discussion is that there as many actions of $\pi_1(K)$ on $\mathbb{R}$ without global fixed points, as actions of $\mathbb{Z}$ on the interval.

2. $\mathsf{BS}(1, 2)$

In the following we consider actions of $\mathsf{BS}(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$ on the real line. We will prove the following.

Theorem 2.1 (Bonatti-Monteverde-Navas-Rivas [8])**.** *Any faithful C^1 action on the closed interval $[0, 1]$ of the group $\mathsf{BS}(1, 2)$ with no global fixed points in $(0, 1)$ is topologically conjugate to the standard affine action of $\mathsf{BS}(1, 2)$ on $\mathbb{R}$. Moreover, the element a corresponding to multiplication by 2, has derivative exactly equal to 2 at its unique interior fixed point.*

Remark 2.2. This result can be extended to actions of abelian-by-cyclic groups Γ_A's.

This will take a relatively long work. The first result we need appears in [17] and simply deals with *continuous* actions.

Proposition 2.3 (Rivas)**.** *Suppose $\mathsf{BS}(1, 2)$ acts on $\mathbb{R}$ with no global fixed points. If b has no fixed point, then the action is semi-conjugate to the standard affine action. If b has a fixed point, then a has no fixed point.*

If b has a fixed point $p \in \mathbb{R}$, then the relation $aba^{-1} = b^2$ implies that all the images $a^k(p)$ are fixed by b. As there is no global fixed point, we must have that $\{a^k(p)\}$ accumulates at $\pm\infty$. Therefore a has no fixed point in $\mathbb{R}$. As for the case of $\mathsf{BS}(1, -1)$, we see that the map b is determined by its behavior on a fundamental domain $I = [p, a(p)]$ (we can assume $a(p) > p$), and by a choice of square roots for all preimages $a^{-k}(I)$. If b has no fixed point, then we need a different strategy. We observe that $\mathsf{BS}(1,2)$ contains a "densely defined" topological flow $b^{\mathbb{Z}[\frac{1}{2}]} = \{b^r; r \in \mathbb{Z}[\frac{1}{2}]\}$.

Lemma 2.4. *The group $b^{\mathbb{Z}[\frac{1}{2}]}$ is semi-conjugate to the "densely defined" flow of translations $T_r : x \mapsto x+r$, $r \in \mathbb{Z}[\frac{1}{2}]$. In particular, there exists an atomless Radon measure ν on $\mathbb{R}$, unique up to scalar multiplication, which is invariant under the action of $b^{\mathbb{Z}[\frac{1}{2}]}$.*

Proof. The action of the abelian group $b^{\mathbb{Z}[\frac{1}{2}]}$ defines an action on the circle $\mathbb{S}^1_b = \mathbb{R}/\langle b\rangle$ (with kernel $b^{\mathbb{Z}}$). Observe that the action of $b^{\mathbb{Z}[\frac{1}{2}]}$ on $\mathbb{R}$ is *free*, that is every element has no fixed point. Thus the same holds for the action on $\mathbb{S}^1_b$. As the group is abelian, the action admits an invariant probability measure μ on $\mathbb{S}^1_b$, which has no atoms (for the action is free). The measure μ lifts to a Radon measure ν on $\mathbb{R}$, which is now invariant under the action of $b^{\mathbb{Z}[\frac{1}{2}]}$. We observe that ν gives the relative translation number

$$\widetilde{\mathsf{rot}}_b(g) = \nu[0, g(0)], \quad g \in b^{\mathbb{Z}[\frac{1}{2}]}$$

with the convention that $\nu[t,s] = -\nu[s,t]$ when $t > s$. Moreover, $\widetilde{\mathsf{rot}}_b$ defines an injective homomorphism from $b^{\mathbb{Z}[\frac{1}{2}]}$ to $\mathbb{R}$. Define the map $F(x) = \nu[0,x]$. We claim that F defines a semi-conjugacy from the action of $b^{\mathbb{Z}[\frac{1}{2}]}$ to $\widetilde{\mathsf{rot}}_b$ (as an action by translations on $\mathbb{R}$). Indeed, we have

$$\begin{aligned} F(g(x)) &= \nu[0, g(x)] = \nu[g(0), g(x)] + \nu[0, g(0)] \\ &= \nu[0,x] + \nu[0,g(0)] = F(x) + \widetilde{\mathsf{rot}}_b(g). \end{aligned}$$

This ends the proof. □

We push the previous proof to obtain a semi-conjugacy to an affine action. The idea is that the relation $aba^{-1} = b^2$ implies that a acts as multiplication by 2 on relative translation numbers. Let us formalize this intuition. The relation $aba^{-1} = b^2$ implies that also the image $a_*\nu$ is invariant, so it must be equal to $\lambda_a\nu$, for some scalar $\nu > 0$. More generally, given $g \in \mathsf{BS}(1,2)$, there exists $\lambda_g > 0$ such that $g_*\nu = \lambda_g\nu$. Moreover, the assignment $g \mapsto \lambda_g$ defines a homomorphism from $\mathsf{BS}(1,2)$ to the multiplicative group $\mathbb{R}^*_+$.

Consider now the map $\psi : \mathsf{BS}(1,2) \to \mathsf{Aff}_+(\mathbb{R})$, defined by

$$\psi_g(x) = \frac{1}{\lambda_g}x + \nu[0, g(0)].$$

Let us verify that ψ is actually a homomorphism. Given $h, g \in \mathsf{BS}(1,2)$, a careful computation gives

$$\begin{aligned}
\psi_{gh}(x) &= \frac{1}{\lambda_{gh}} x + \nu[0, gh(0)] \\
&= \frac{1}{\lambda_g}\left(\frac{1}{\lambda_h} x + \nu[g^{-1}(0), h(0)]\right) \\
&= \frac{1}{\lambda_g}\left(\frac{1}{\lambda_h} x + \nu[g^{-1}(0), 0] + \nu[0, h(0)]\right) \\
&= \frac{1}{\lambda_g}\left(\frac{1}{\lambda_h} x + \lambda_g \nu[0, g(0)] + \nu[0, h(0)]\right) \\
&= \psi_g \psi_h(x).
\end{aligned}$$

Finally, as before, we prove that $F(x) = \nu[0, x]$ defines a semi-conjugacy from the initial action of $\mathsf{BS}(1,2)$ to the one defined by ψ:

$$\begin{aligned}
F(g(x)) &= \nu[0, g(x)] = \nu[g(0), g(x)] + \nu[0, g(0)] \\
&= \frac{1}{\lambda_g} \nu[0, x] + \nu[0, g(0)] = \psi_g(F(x)).
\end{aligned}$$

This is what we wanted to prove.

3. C^1 ACTIONS

From now on we will be interested in C^1 actions and for this reason we pass from $\mathbb{R}$ to $[0,1]$, as differentiability at endpoints is crucial.

Remark 3.1. The chain rule guarantees that if G acts C^1 on the interval $[0,1]$, then every commutator has derivative equal to 1 at both the endpoints $0, 1$.

However, up to topological conjugacy, one can assume that this is the case for every element in the group: one can indeed conjugate the action using a homeomorphism which is locally of the form $\mathsf{sgn}(x)\exp(-1/|x|)$ on a neighborhood of the endpoints. This trick goes back to Muller [16] and Tsuboi [19].

Proposition 3.2 (Cantwell-Conlon [10]; Guelman-Liousse [11])**.** *Suppose* $\mathsf{BS}(1,2)$ *acts on* $[0,1]$ *with no global fixed points in* $(0,1)$ *and such that* b *has a fixed point in* $(0,1)$*. Then the action cannot be* C^1*. In particular any* C^1 *action of* $\mathsf{BS}(1,2)$ *on the interval* $[0,1]$ *is semi-conjugate to the standard affine action.*

Proof. We have seen that if p is a fixed point of b, then the interval $I = (p, a(p))$ is a fundamental domain for the action of a and is fixed by b. As the intervals $a^k(I)$, $k \in \mathbb{Z}$, are all preserved by b, then b must have derivative equal to 1 at the endpoints $0, 1$. Actually, using Remark 3.1, we can assume something stronger. Choose $\varepsilon > 0$ so that $(1-\varepsilon)^3 > \frac{1}{2}$. Then we can assume that there exists k_0 such that for every $|k| \geq k_0$, the derivatives of $a^{\pm 1}, b^{\pm 1}$ are ε-close to 1 on every interval $a^k(I)$. We write $I' = a^{-k_0}(I)$.

Choose a point $x \in I'$ which is not fixed by b. Consider the interval $J = (x, b(x))$, which has the property that the images $b^n(J)$, $n \in \mathbb{Z}$, are pairwise disjoint and contained in I'.

Fix $m \in \mathbb{N}$. Given $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$, we consider the element

$$\begin{aligned} B(\underline{\epsilon}) &= a^m (b^{-\epsilon_m} a^{-1})(b^{-\epsilon_{m-1}} a^{-1}) \cdots (b^{-\epsilon_1} a^{-1}) \\ &= \prod_{i=1}^{m} a^i b^{-\epsilon_i} a^{-i} \end{aligned}$$

which is equal to $b^{-R(\underline{\epsilon})}$, with $R(\underline{\epsilon}) = \sum_{i=1}^m \epsilon_i 2^i$.

(1) With respect to the generating system $\{a, b\}$, $B(\underline{\epsilon})$ belongs to the ball of radius $3m$.

(2) For any $x \in J$, the point $B(\underline{\epsilon})(x) = b^{-R(\underline{\epsilon})}(x)$ lies on the left of x. Moreover this holds for any of the (at most) first $2m$ intermediate images of x, while the last m iterations of a bring x back into I'. Thus, by our condition on the derivatives, we get the bound

$$(1-\varepsilon)^{3m} \leq B(\underline{\epsilon})'(x).$$

We deduce that

$$(1-\varepsilon)^{3m} |J| \leq |B(\underline{\epsilon})(J)|.$$

(3) For different sequences $\underline{\epsilon} \in \{0,1\}^m$, the elements $B(\underline{\epsilon})$ are all different and the intervals $B(\underline{\epsilon})(J)$ are pairwise disjoint. (We are simply writing the integer R in the dyadic basis.) Thus the previous estimate gives

$$2^m (1-\varepsilon)^{3m} |J| \leq \sum_{\underline{\epsilon} \in \{0,1\}^m} |B(\underline{\epsilon})(J)| \leq 1 = |[0,1]|.$$

After our choice of $\varepsilon > 0$, this gives a contradiction for large values of m. □

A similar argument yields the following:

Proposition 3.3 (Bonatti-Monteverde-Navas-Rivas [8])**.** *Any C^1 action of $\mathsf{BS}(1,2)$ on the interval $[0,1]$ without global fixed points is topologically conjugate to the standard affine action.*

Proof. From the previous proof, we already know that any such action is semi-conjugate to the standard one. To complete the proof, we need to show that the action is minimal. That is, we want to prove that there is no wandering interval for the "topological flow" $b^{\mathbb{Z}[\frac{1}{2}]}$. One can simply repeat the previous proof. □

Proposition 3.4 (Bonatti-Monteverde-Navas-Rivas [8])**.** *Any C^1 action of $\mathsf{BS}(1,2)$ on the interval $[0,1]$ without global fixed points has the property that at the unique fixed point $p \in (0,1)$ of a, $a'(p) = 2$.*

Proof. We proceed as before, but working also on a neighborhood of p. As before, we suppose that on a given neighborhood of the endpoints of the interval, the derivatives of the generators $a^{\pm 1}, b^{\pm 1}$ are ε-close to 1 (we shall choose ε at the very end of the argument). Let I_0 be the interval $[p, b(p)]$. Fix $m \in \mathbb{N}$, and set $I_m = a^{-m}(I_0) = [p, a^{-m}b(p)] = [p, b^{1/2^m}(p)]$.

Suppose first $a'(p) < 2$. Fix $\delta > 0$ and a neighborhood of p such that $(a^{-1})'(x) > \frac{1}{2} + \delta$ for any x in the neighborhood. Then there exists $C > 0$ such that

$$|I_m| = |a^{-m}(I_0)| > C \left(\frac{1}{2} + \delta \right)^m |I_0|.$$

Given a sequence $\underline{\epsilon} \in \{0,1\}^m$, we consider now the element

$$\begin{aligned} b(\underline{\epsilon}) &= a^{-m} (b^{\epsilon_m} a) \cdots (b^{\epsilon_1} a) \\ &= \prod_{i=1}^{m} a^{-i} b^{\epsilon_i} a^i, \end{aligned}$$

which is equal to $b^{r(\underline{\epsilon})}$, with $r(\underline{\epsilon}) = \sum_{i=1}^{m} \epsilon_i 2^{-i}$ (which is a dyadic rational in $[0, \frac{2^m-1}{2^m}]$ of the form $l/2^m$). We observe the following.

(1) With respect to the generating system $\{a, b\}$, $b(\underline{\epsilon})$ belongs to the ball of radius $3m$.
(2) For any $x \in I_m$, the point $b(\underline{\epsilon})(x) = b^{r(\underline{\epsilon})}(x)$ lies on the right of x. Moreover this holds for any of the (at most) first $2m$ intermediate images of x, while the last m iterations of a bring x back, but not past I_m.
(3) For different sequences $\underline{\epsilon} \in \{0,1\}^m$, the elements $b(\underline{\epsilon})$ are all different and the intervals $b(\underline{\epsilon})(I_m^\circ)$ are pairwise disjoint, covering I_0.

Consider some large $N \in \mathbb{N}$ so that the interval $J_m = b^N(I_m)$ is contained in the neighborhood of 1 on which we have the good control of the derivatives of the generators. Take $D = \min_{[0,1]}(b^N)'$, so that

$$|J_m| \geq CD\left(\frac{1}{2}+\delta\right)^m |I_0|. \tag{1}$$

The previous considerations on the elements $b(\underline{\epsilon})$ hold also when considering their restrictions to J_m, for they all commute with b^N. However now we can use the control on derivatives: for any $x \in J_m$, we have

$$(1-\varepsilon)^{3m} \leq (b(\underline{\epsilon}))'(x),$$

therefore

$$(1-\varepsilon)^{3m}|J_m| \leq |b(\underline{\epsilon})(J_m)|.$$

Summing over all possible sequences $\underline{\epsilon} \in \{0,1\}^m$, gives

$$2^m(1-\varepsilon)^{3m}|J_m| \leq \sum_{\underline{\epsilon}\in\{0,1\}^m} |b(\underline{\epsilon})(J_m)| \leq 1.$$

Finally, (1) gives

$$2^m(1-\varepsilon)^{3m}CD\left(\frac{1}{2}+\delta\right)^m |I_0| \leq 1.$$

If we choose $\varepsilon > 0$ so that $2(1-\varepsilon)^3\left(\frac{1}{2}+\delta\right) > 1$, we get the desired contradiction.

Assuming $a'(p) > 2$, we can repeat similar arguments, working with inequalities on the reversed sides. The details are left to the reader. □

4. The broken dyadic affine group

The "broken" dyadic affine group G_2 has been introduced in [7] in order to find an obstruction for the existence of C^1 actions. In fact, there are only a few known obstructions, the most celebrated being given by Thurston's stability theorem [18], which states that any group of C^1 diffeomorphisms of the closed interval is *locally indicable*, that is, every finitely generated subgroup surjects to $\mathbb{Z}$.

Our group G_2 is built upon the dyadic affine group $\mathsf{BS}(1,2) = \langle D, T \mid DTD^{-1} = D^2\rangle$, with the addition of "half" of the scalar multiplication D: for this we define $H : \mathbb{R} \to \mathbb{R}$ by setting

$$H(x) = \begin{cases} x & \text{if } x < 0, \\ 2x & \text{if } x \geq 0. \end{cases}$$

It is also useful to define $K = DH^{-1} = H^{-1}D$, which corresponds to the "negative half". Observe that H, K generate an abelian group of rank 2, which contains D. The group G_2 is the subgroup of $\mathsf{Homeo}_+(\mathbb{R})$ generated by D, T, H. By construction, G_2 is a subgroup of the group $\mathsf{PL}_+(\mathbb{R})$, of *piecewise affine homeomorphisms*. Although the explicit definition of G_2, we do not know a presentation of this group.

Let us see an easy consequence of Theorem 2.1.

Proposition 4.1. *The standard action of* G_2 *on* $\mathbb{R}$ *cannot be topologically conjugate to a* C^1 *action on the closed interval* $[0,1]$.

Proof. Suppose such a conjugacy existed, and write $d, t, h, \ldots$ for the corresponding conjugated diffeomorphisms of $[0,1]$ coming from $D, T, H, \ldots$ Denote by p the unique interior fixed point of d. Theorem 2.1 tells that $d'(p) = 2$. As d and h coincide on the right of p, this gives $h'(p) = d'(p) = 2$. On the other side, h is the identity on the left of p, so we must have $h'(p) = 1$. Contradiction. $\square$

The main result of [7] is the following:

Theorem 4.2 (Bonatti-Lodha-T)**.** *The group* G_2 *has no faithful* C^1 *action on the closed interval* $[0,1]$*. More precisely, for any representation* $\rho : G_2 \to \mathsf{Diff}_+([0,1])$*, the commutator* $[T, HDH^{-1}]$ *belongs to the kernel of* ρ.

The proof is way more involved, and we do not enter into the details here.

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