Ping-Pong Lemma – Old and New

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Based on joint work with

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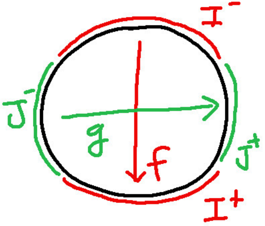
Classical Ping-Pong



$$\begin{cases} f(\mathbb{S}^1 \setminus I^-) = \overline{I^+} \\ f^{-1}(\mathbb{S}^1 \setminus I^+) = \overline{I^-} \\ g(\mathbb{S}^1 \setminus J^-) = \overline{J^+} \\ g^{-1}(\mathbb{S}^1 \setminus J^+) = \overline{J^-} \end{cases}$$

1. **Ping-Pong Lemma:** f, g generate a rank-2 free group: $\langle f, g\rangle \cong \pi_1(\ )$ (*faithful action*)
2. the C^0 semi-conjugacy class of the action on $\mathbb{S}^1$ is uniquely determined by the **cyclic ordering** of the 4 intervals and the **inclusion relations**: *up to change of coordinates, it is the action* $\pi_1(\ ) \curvearrowright \mathbb{S}^1 \cong \partial\widetilde{\ }$

$\rightsquigarrow$ extended in recent works by Matsumoto, Mann–Rivas (–Malicet–T.)

Next step: virtually free groups

Why?

- They naturally occur as groups of circle diffeomorphisms:

Theorem (Ghys)

*Let G be a finitely generated group of real-analytic circle diffeomorphisms with invariant Cantor set, then G is **virtually free**.*

(generalization by Alvarez–Filimonov–Kleptsyn–Malicet–Meniño–Navas–T.)

- **Deroin–Kleptsyn–Navas:** ping-pong partition for finite-index free subgroups
- **Main motivation: foliation theory** Dippolito (1978) conjectured that the action of such a group must be C^0-conjugate to a piecewise-linear action

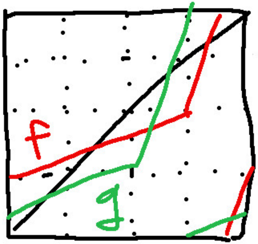
Solving Dippolito conjecture

Conjecture (Dippolito, particular case)

Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated group with invariant Cantor set. Then G is C^0-conjugate to a PL action.

For actions admitting ping-pong partitions it is very easy to make them PL:

1. realize all the inclusions relations between intervals of the partition using PL homeomorphisms
2. the ping-pong lemma says that the action is faithful
3. two actions with the same ping-pong partition must be semi-conjugate (in case of real-analytic regularity one can actually improve this to conjugacy)



Problem: Find a good notion of ping-pong partition for virtually free groups, so that:

1. If the action admits a ping-pong partition then the action is faithful
2. The ping-pong partition consists of finitely many data which uniquely determine the semi-conjugacy class of the action

Note: Ping-pong partition, even in the classical case of free groups, depends on the free generating set

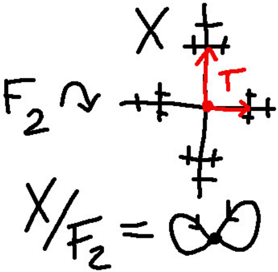
Alternative way of thinking about it (classical)

a group G is free if and only if it admits a free action on a tree

$\rightsquigarrow$ choice of free generating set corresponds to:

1. free action of G on a (locally finite) tree X
2. connected fundamental domain $T \subset X$

Indeed, this gives an identification $\pi_1(X/G) \cong G$ with marked free generating set



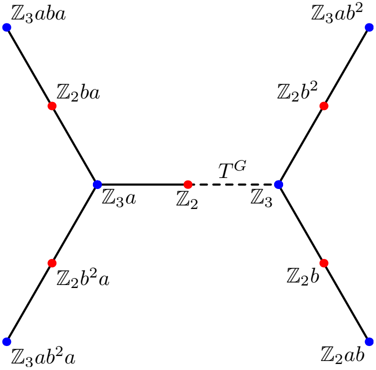
Theorem (Karrass–Pietrowki–Solitar)

A group G is virtually free if and only if it admits a proper action on a tree.

(*Proper action:* finite stabilizers)

Marking of a virtually free group: proper action on a tree + connected fundamental domain

Example: $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$



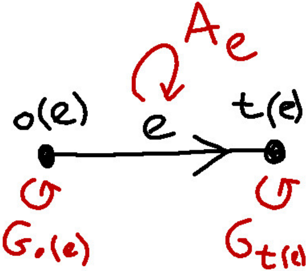
Bass–Serre theory

A marking determines a presentation of the group G as the **fundamental group of a graph of groups**: let $\overline{X} := X/G = (V, E)$ be the graph

$$G \cong \pi_1(\overline{X}; G_v, A_e) = \left\langle G_v, E \;\middle|\; \begin{array}{ll} \mathrm{rel}(G_v) & \text{for } v \in V, \\ \overline{e} = e^{-1} & \text{for } e \in E, \\ e = id & \text{for } e \in E_T, \\ e^{-1}\alpha_e(g)e = \omega_e(g) & \text{for } e \in E, g \in A_e \end{array} \right\rangle.$$

where

- G_v is the vertex group: stabilizer of the vertex v *(lift $v \in \overline{X}$ to the vertex in the fundamental domain $T \subset X$)*
- A_e is the edge group, which embeds in both $G_{o(e)}$ (initial vertex) and $G_{t(e)}$ (target vertex)



Virtually free groups acting on the circle

In the case of marked virtually free groups, all vertex groups are finite.
If moreover the group acts on the circle, vertex groups must be cyclic finite.

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From these considerations, we can prove the following

Theorem (AAMMT)

Let $G \subset \mathrm{Homeo}_+(\mathbb{S}^1)$ be a virtually free group. Then there exists a normal free subgroup with finite cyclic quotient. Reciprocally, every group G

$$1 \to \mathrm{Free} \to G \to \mathbb{Z}_m \to 0$$

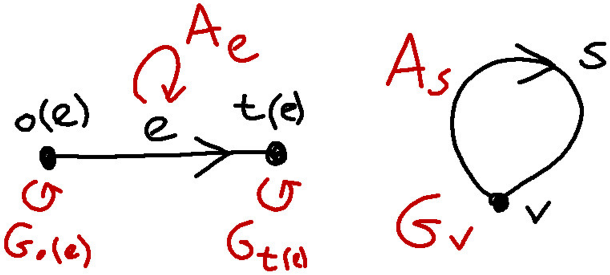
is realized as subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$.

Generalized ping-pong lemma

Basic cases of fundamental groups of graphs of groups are

1. amalgamated free products
2. HNN extensions

(Fundamental groups of graphs of groups: finitely many such operations)



In both situations **Fenchel–Nielsen, Maskit** proved a ping-pong lemma

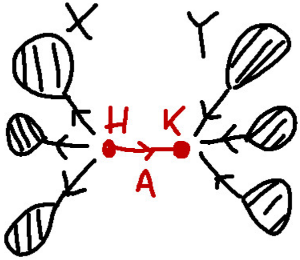
Ping-pong lemma for amalgamated free products

Proposition (Fenchel–Nielsen)

Let $G = H *_A K$ be an amalgamated free product and consider an action on a set $G \curvearrowright \Omega$. Assume that

1. *there are disjoint subsets $X, Y \subset \Omega$ such that:*
2. $A(X) = X$ *and* $A(Y) = Y$;
3. $(H \setminus A)(Y) \subset X$ *and* $(K \setminus A)(X) \subset Y$;
4. *the complement* $X \setminus (H \setminus A)(Y)$ *is not empty;*
5. *A acts faithfully.*

Then the action $G \curvearrowright \Omega$ is faithful.



Ping-pong lemma for HNN extensions

Proposition (Fenchel–Nielsen)

Let $G = H*_A$ be an HNN extension and consider an action on a set $G \curvearrowright \Omega$.
Assume that

1. *there are disjoint subsets $X, Z^+, Z^- \subset \Omega$ such that:*
2. $s(Z^+ \cup X) \subset Z^+$ *and* $s^{-1}(Z^- \cup X) \subset Z^-$;
3. $\alpha_s(A)(Z^+) = Z^+$ *and* $\omega_s(A)(Z^-) = Z^-$;
4. $(H \setminus \alpha_s(A))(Z^+) \subset X$ *and* $(H \setminus \omega_s(A))(Z^-) \subset X$;
5. *the complement* $X \setminus ((H \setminus \alpha_s(A))(Z^+) \cup (H \setminus \omega_s(A))(Z^-))$ *is not empty;*
6. $\alpha_s(A)$ *and* $\omega_s(A)$ *act faithfully.*

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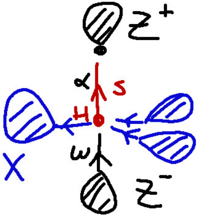

Generalized ping-pong lemma

Theorem (AAMMT)

Let $G = \pi_1(\overline{X}; G_v, A_e)$ be the fundamental group of a graph of groups and consider an action on a set $G \curvearrowright \Omega$.
If the action satisfies a list of 10 conditions (of similar flavor), then the action $G \curvearrowright \Omega$ is faithful.

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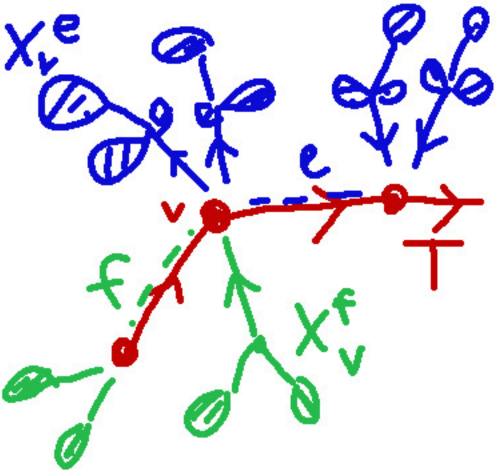
3. The result is valid for every fundamental group of a graph of groups *(not only virtually free groups)*.
4. If you know a place in the literature where this has been done, I'm very interested!!!



X_v^e
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X_v^f

Ping-pong partitions of the circle

Theorem (AAMMT)

*There is a **good notion of ping-pong partition** of the circle for actions of virtually free groups, which consists of finitely many combinatorial data, and which uniquely determines the semi-conjugacy class of the action.*

Theorem (AAMMT)

Every finitely generated group $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ of real-analytic circle diffeomorphisms admits a ping-pong partition.

ping-pong partition: previous 10 conditions + 5 more (finitely many connected components, Markovian dynamics,...)

Thank you!